Sec 13.3: Arc Length and Curvature

DEF. Suppose that a curve has a vector equation $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle,\ a \le t \le b$, where f', g', and h' are continuous. If the curve is traversed exactly once as t increases from a to b.

Its arc length is defined to be $L = \int_a^b ||\mathbf{r}'(t)||dt$.

Ex1. Find the arc length of the curve with vector equation $\mathbf{r}(t) = \langle t^2/2,\ \sqrt{2}t,\ \ln t\rangle$ in $\mathbb{R}^3$, defined on $[1, 2]$.

$r'(t) = \langle t, \sqrt{2}, 1/t\rangle$

$$\text{arclength} = \int_1^2 ||r'(t)||\, dt = \int_1^2 \sqrt{t^2 + 2 + 1/t^2}\, dt = \int_1^2 \sqrt{(t + 1/t)^2}\, dt$$

$$= \int_1^2 t + \frac{1}{t}\, dt = \left(\frac{t^2}{2} + \ln|t|\right)\Big|_1^2$$

$$= 2 + \ln(2) - \left(\frac{1}{2} + \cancel{\ln(1)}^{\,0}\right)$$

$$= \boxed{\frac{3}{2} + \ln(2)}$$

A parametrization $\mathbf{r}(t)$ is called smooth on an interval I if $\mathbf{r}'$ is continuous and $\mathbf{r}'(t) \ne \vec{0}$ on I. A curve is called smooth if it has a smooth parametrization. A smooth curve has no sharp corners or cusps; when the tangent vector turns, it does so continuously.

DEF. Let C be a smooth curve defined by the parametrization $\mathbf{r}(t)$. The unit tangent vector at $\mathbf{r}(t)$ is denoted by $\mathbf{T}(t)$ and defined by

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{||\mathbf{r}'(t)||}$$

This vector indicates the direction of the curve.

Ex2. Let $\mathbf{r}(t) = \langle \cos e^t, \sin e^t, e^t\rangle$. Find the vector $\mathbf{T}(t)$. (Find the unit tangent vector at time t)

$r'(t)\ \langle -e^t \sin(e^t), e^t\cos(e^t), e^t\rangle$

$r'(t) = e^t \langle -\sin(e^t), \cos(e^t), 1\rangle$

now $||r'(t)|| = |e^t|\ ||\langle -\sin(e^t), \cos(e^t), 1\rangle||$

$$= e^t\sqrt{\sin^2(e^t) + \cos^2(e^t) + 1} = e^t\sqrt{1+1} = \sqrt{2}e^t$$

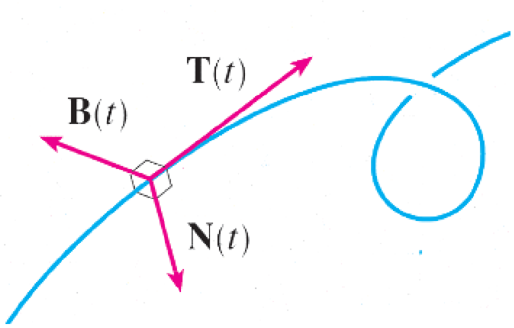
then $T(t) = \frac{1}{||r'(t)||} r'(t) = \frac{1}{\sqrt{2}\cancel{e^t}} \cancel{e^t} \langle -\sin(e^t), \cos(e^t), 1\rangle$

so $\boxed{T(t) = \frac{1}{\sqrt{2}} \langle -\sin(e^t), \cos(e^t), 1\rangle}$

Let C be a smooth curve with parametrization $\mathbf{r}(t)$ and let $\mathbf{T}(t)$ be the unit tangent vector. If $\mathbf{T}'(t) \neq \vec{0}$, then the vector $\mathbf{N}(t)$ given by

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{||\mathbf{T}'(t)||}$$

is well defined.



$\mathbf{N}(t)$ is the **principal unit normal vector** at $\mathbf{r}(t)$, this vector is perpendicular to $\mathbf{T}(t)$, **Why?**

$\|T(t)\| = 1 \Rightarrow T(t) \perp T'(t) \Rightarrow T(t) \perp \dfrac{T'(t)}{\|T'(t)\|}$

Let a third vector that is perpendicular to both $\mathbf{T}(t)$ and $\mathbf{N}(t)$ be defined by $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$; $\mathbf{B}(t)$ is called the **binormal vector** at $\mathbf{r}(t)$.Together, $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$ form a right-handed system of orthogonal unit vectors moving along the path.

Ex3. Show that $\mathbf{B}(t)$ is actually a unit vector.

$$\|B(t)\| = \|T(t) \times N(t)\| = \underbrace{\|T(t)\|}_{(1)}\ \underbrace{\|N(t)\|}_{(1)}\ \underbrace{\sin\theta}_{(1)} \qquad (\text{where } \theta = \pi/2)$$

$$= (1)\ (1)\ (1)$$

$$= 1$$

Ex4. Consider the vector-valued function $\mathbf{r}(t) = \langle 3, t+1, t^2 \rangle$. $\Rightarrow r'(t) = \langle 0, 1, 2t \rangle$
(a) Find the unit normal vector $\mathbf{N}(t)$. $\Rightarrow \|r'(t)\| = \sqrt{0^2 + 1^2 + 4t^2}$

Now, $T(t) = \dfrac{1}{\|r'(t)\|} r'(t) = \dfrac{1}{\sqrt{1+4t^2}} \langle 0, 1, 2t \rangle$

$\dfrac{d}{dt}(1+4t^2)^{-1/2} = \dfrac{-1(8t)}{2(1+4t^2)^{3/2}}$

9d rule

then, $T'(t) = \dfrac{-4t}{(1+4t^2)^{3/2}} \langle 0, 1, 2t \rangle + \dfrac{1}{\sqrt{1+4t^2}} \langle 0, 0, 2 \rangle \dfrac{(1+4t^2)}{(1+4t^2)}$

$= \dfrac{1}{(1+4t^2)^{3/2}} \left(\langle 0, -4t, -8t^2 \rangle + \langle 0, 0, 2+8t^2 \rangle \right) = \dfrac{1}{(1+4t^2)^{3/2}} \langle 0, -4t, 2 \rangle$

thus, $\|T'(t)\| = \dfrac{1}{(1+4t^2)^{3/2}} \|\langle 0, -4t, 2 \rangle\| = \dfrac{1}{(1+4t^2)^{3/2}} \sqrt{16t^2+4} = \dfrac{2\sqrt{4t^2+1}}{(1+4t^2)^{3/2}}$

so, $\|T'(t)\| = \dfrac{2}{1+4t^2}$

therefore, $N(t) = \dfrac{1}{\|T'(t)\|} T'(t) = \dfrac{(1+4t^2)}{2} \cdot \dfrac{1}{(1+4t^2)^{3/2}} \langle 0, -4t, 2 \rangle = \dfrac{1}{\sqrt{1+4t^2}} \langle 0, -2t, 1 \rangle$

(b) Find the binormal vector at $t = 1$. $B(t) = T(t) \times N(t)$

$B(1) = T(1) \times N(1) = \left(\dfrac{1}{\sqrt{5}} \langle 0, 1, 2 \rangle \right) \times \left(\dfrac{1}{\sqrt{5}} \langle 0, -2, 1 \rangle \right) = \dfrac{1}{5} \left(\langle 0, 1, 2 \rangle \times \langle 0, -2, 1 \rangle \right)$

then $B(1) = \dfrac{1}{5} \langle 5, 0, 0 \rangle = \langle 1, 0, 0 \rangle$

Normal Plane and Osculating Plane

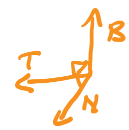
Let $\mathbf{r}(t)$ be a smooth parametrization of the curve C and let $P = \mathbf{r}(t_0)$ be a point on the curve C. We have the following definitions:

- The **normal plane** of C at the point P is the plane that passes through the point P and is perpendicular to the vector $\mathbf{T}(t_0)$. Note that this plane contains the vectors $\mathbf{N}(t_0)$ and $\mathbf{B}(t_0)$.
- The **osculating plane** of C at the point P is the plane that passes through the point P and is perpendicular to the vector $\mathbf{B}(t_0)$. It is the plane that comes closest to containing the part of the curve near the point P. Note that this plane contains the vectors $\mathbf{T}(t_0)$ and $\mathbf{N}(t_0)$.

see example 4

Ex5. Find equations for the normal plane and the osculating plane to the path $\mathbf{r}(t) = \langle 3, t+1, t^2 \rangle$ at the point $P(3, 2, 1)$.

At the point $(3,2,1)$, the value of t is 1.



Normal plane: point: $(3,2,1)$

•) a perpendicular vector is $T(1) = \frac{1}{\sqrt{5}} \langle 0, 1, 2 \rangle$

Eq. of normal plane:

$$\langle x-3, y-2, z-1 \rangle \cdot \langle 0, \tfrac{1}{\sqrt{5}}, \tfrac{2}{\sqrt{5}} \rangle = 0$$

$$\frac{1}{\sqrt{5}}(y-2) + \frac{2}{\sqrt{5}}(z-1) = 0 \Rightarrow y-2+2(z-1)=0$$

$$\boxed{y + 2z = 4}$$

Osculating plane: •) point: $(3,2,1)$

•) a perpendicular vector is $B(1) = \langle 1, 0, 0 \rangle$

Eq of osculating plane:

$$\langle x-3, y-2, z-1 \rangle \cdot \langle 1, 0, 0 \rangle = 0$$

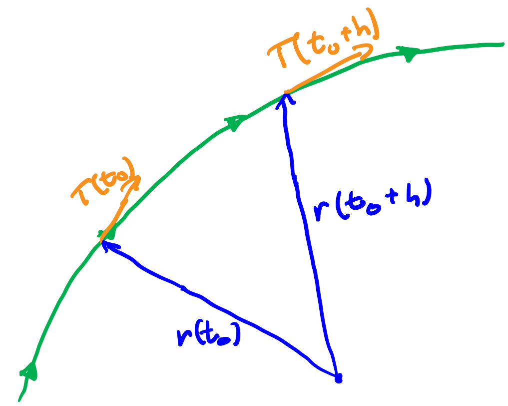
$$1(x-3) + 0 + 0 = 0$$

$$\boxed{x = 3}$$

Curvature: The curvature at the point $\mathbf{r}(t)$ is a measure of how quickly the curve changes direction at that point. The curvature at $\mathbf{r}(t)$ is defined by

$$k(t) = \frac{||\mathbf{T}'(t)||}{||\mathbf{r}'(t)||}$$

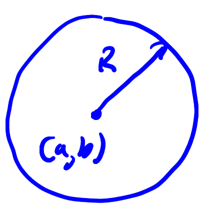
Geometric Idea:



$$\lim_{h\to 0} \frac{||T(t_0+h) - T(t_0)|| \left(\frac{1}{|h|}\right)}{||r(t_0+h) - r(t_0)|| \left(\frac{1}{|h|}\right)} = \lim_{h\to 0} \frac{\left|\left| \frac{T(t_0+h)-T(t_0)}{h} \right|\right|}{\left|\left| \frac{r(t_0+h)-r(t_0)}{h} \right|\right|}$$

$$k(t_0) = \frac{||T'(t_0)||}{||r'(t_0)||}$$

Ex6. Show that the curvature of a circle of radius R is $1/R$.



$$\vec{r}(t) = \langle a + R\cos t,\ b + R\cos t,\ 0 \rangle$$

$$\bullet)\ \vec{r}\,'(t) = \langle -R\sin t,\ R\cos t,\ 0 \rangle \Rightarrow ||r'(t)|| = \sqrt{R^2\sin^2 t + R^2\cos^2 t}$$

$$\Rightarrow \boxed{||r'(t)|| = R}$$

$$\bullet)\ T(t) = \frac{1}{||r'(t)||} \cdot r'(t) = \frac{1}{R} \langle -R\sin t, R\cos t, 0 \rangle = \langle -\sin t, \cos t, 0 \rangle.$$

$$\Rightarrow T'(t) = \langle -\cos t, -\sin t, 0 \rangle \Rightarrow ||T'(t)|| = \sqrt{\cos^2 t + \sin^2 t}$$

$$\Rightarrow \boxed{||T'(t)|| = 1}$$

$$\text{so} \quad \boxed{k(t) = \frac{1}{R}.}$$

The formula given by the following theorem is often more convenient to apply.

Theorem The curvature of the curve given by the vector function $\mathbf{r}(t)$ is

$$k(t) = \frac{||\mathbf{r}'(t) \times \mathbf{r}''(t)||}{||\mathbf{r}'(t)||^3}.$$

Ex7. Find the curvature of $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$ at $t = 0$.

$r'(t) = \langle 1, 2t, 3t^2 \rangle, \quad r''(t) = \langle 0, 2, 6t \rangle.$

$$k(0) = \frac{||r'(0) \times r''(0)||}{||r'(0)||^3} = \frac{||\langle 1,0,0\rangle \times \langle 0,2,0\rangle||}{||\langle 1,0,0\rangle||^3} = \frac{||\langle 0,0,2\rangle||}{1} = 2$$

Ex8. Find the curvature of the parabola $y = x^2$ at the point $(2, 4)$.

Let $x = t, \ y = t^2$

$r(t) = \langle t, t^2, 0 \rangle$

$r'(t) = \langle 1, 2t, 0 \rangle$

$r''(t) = \langle 0, 2, 0 \rangle$

when $t = 2$, we have the point $(x, y, z) = (2, 4, 0)$.

then $$k(2) = \frac{||r'(2) \times r''(2)||}{||r'(2)||^3} = \frac{||\langle 1,4,0\rangle \times \langle 0,2,0\rangle||}{||\langle 1,4,0\rangle||^3} = \frac{||\langle 0,0,2\rangle||}{(\sqrt{1+16})^3} = \frac{2}{\sqrt{17}^3}$$

Extra Question: $k(t) = ?$

$$k(t) = \frac{||r'(t) \times r''(t)||}{||r'(t)||^3} = \frac{||\langle 1,2t,0\rangle \times \langle 0,2,0\rangle||}{||\langle 1,2t,0\rangle||^3} = \frac{||\langle 0,0,2\rangle||}{(\sqrt{1+4t^2})^3} = \frac{2}{(\sqrt{1+4t^2})^3}$$

when $t \to \infty$, $k(t) \to 0$

Sec 13.4 Motion in Space: Velocity and Acceleration

Given a path $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$, we define the following.

- Velocity vector: $\mathbf{v}(t) := \mathbf{r}'(t)$
- Speed: $s(t) := ||\mathbf{v}(t)||$ $= ||r'(t)||$
- Acceleration vector: $\mathbf{a}(t) := \mathbf{v}'(t)$ $= r''(t)$

Ex1. The acceleration vector of an object is $\mathbf{a}(t) = 4t\mathbf{i} + 6t\mathbf{j} + \mathbf{k}$, for $t \geq 0$. The initial velocity and position are: $\mathbf{v}(0) = \mathbf{i} - \mathbf{j} + \mathbf{k}$ and $\mathbf{r}(0) = \langle 1, 0, 0\rangle$. Find the velocity and the position of the object at time t.

$a(t) = \langle 4t, 6t, 1\rangle$

then $v(t) = \langle 2t^2, 3t^2, t\rangle + \vec{C}$ where $\vec{C} = \langle c_1, c_2, c_3\rangle$

but $\underbrace{v(0)}_{} = \langle 1, -1, 1\rangle$

$\langle 2(0)^2, 3(0)^2, 0\rangle + \vec{C} = \langle 1, -1, 1\rangle \Rightarrow \vec{0} + \vec{C} = \langle 1, -1, 1\rangle$

$\vec{C} = \langle 1, -1, 1\rangle$

thus, $\boxed{v(t) = \langle 2t^2+1, 3t^2-1, t+1\rangle}$

then, $r(t) = \left\langle \frac{2}{3}t^3 + t, t^3 - t, \frac{t^2}{2} + 1\right\rangle + \vec{D}$, where $\vec{D} = \langle d_1, d_2, d_3\rangle$

but $\underbrace{r(0)}_{} = \langle 1, 0, 0\rangle$

$\langle \frac{2}{3}(0) + 0, 0 - 0, 0 + 0\rangle + \vec{D} = \langle 1, 0, 0\rangle \Rightarrow 0 + \vec{D} = \langle 1, 0, 0\rangle$

$\Rightarrow \vec{D} = \langle 1, 0, 0\rangle$

so $\boxed{r(t) = \left\langle \frac{2}{3}t^3 + t + 1, t^3 - t + 0, \frac{t^2}{2} + t + 0\right\rangle}$

$$\bullet\ T = \frac{r'}{\|r'\|} = \frac{v}{\|v\|} \Rightarrow \boxed{v = \|v\|\,T}$$

$$\bullet\ N = \frac{T'}{\|T'\|} \Rightarrow \boxed{T' = \|T'\|\,N}$$

$$\bullet\ k = \frac{\|T'\|}{\|r'\|} \Rightarrow \boxed{\|T'\| = k\,\|r'\|}$$

Tangential and Normal Components of Acceleration:

Note that $\mathbf{v} = ||\mathbf{v}||\mathbf{T}$, then by product rule we have

$$\mathbf{v}' = \{||\mathbf{v}||\}'\ \mathbf{T} + ||\mathbf{v}||\ \mathbf{T}'$$

Since $\{||\mathbf{v}||\}' = \dfrac{\mathbf{v}\cdot\mathbf{v}'}{||\mathbf{v}||}$ and $\mathbf{T}' = ||\mathbf{T}'||\ \mathbf{N}$ we have

$$\mathbf{a} = \frac{\mathbf{v}\cdot\mathbf{v}'}{||\mathbf{v}||}\ \mathbf{T} + \{||\mathbf{v}||\ ||\mathbf{T}'||\}\ \mathbf{N}$$

So the acceleration vector lies in the osculating plane. Moreover, since $\mathbf{v} = \mathbf{r}'$ and $||\mathbf{T}'|| = k||\mathbf{r}'||$ we have

$$\mathbf{a} = \frac{\mathbf{r}'\cdot\mathbf{r}''}{||\mathbf{r}'||}\ \mathbf{T} + \{k||\mathbf{r}'||^2\}\ \mathbf{N}$$

$$\bullet\ k = \frac{\|r' \times r''\|}{\|r'\|^3}$$

where k is the curvature. This yields the following definitions:

- Tangential component: $a_T = \dfrac{\mathbf{r}'\cdot\mathbf{r}''}{||\mathbf{r}'||}$
- Normal component: $a_N = k||\mathbf{r}'||^2 = \dfrac{||\mathbf{r}'\times\mathbf{r}''||}{||\mathbf{r}'||}$

Ex2. A particle moves with position function $\mathbf{r}(t) = \langle t^2, t^2, t^3\rangle$. Find the tangential and normal components of acceleration.

$$r'(t) = \langle 2t, 2t, 3t^2\rangle = t\langle 2, 2, 3t\rangle$$

$$r''(t) = \langle 2, 2, 6t\rangle = 2\langle 1, 1, 3t\rangle$$

$$\bullet)\ a_T = \frac{r'\cdot r''}{\|r'\|} = \frac{\langle 2t, 2t, 3t^2\rangle\cdot\langle 2, 2, 6t\rangle}{\|\langle 2t, 2t, 3t^2\rangle\|} = \frac{8t + 18t^3}{\sqrt{8t^2 + 9t^4}} = \frac{\cancel{t}(8 + 18t^2)}{\cancel{t}\sqrt{8 + 9t^2}}$$

$$\text{so, } a_T = \frac{8 + 18t^2}{\sqrt{8 + 9t^2}}$$

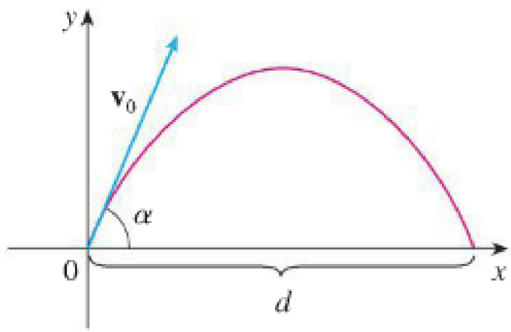
$$\bullet)\ a_N = \frac{\|r' \times r''\|}{\|r'\|} = \frac{\|\langle 2t, 2t, 3t^2\rangle \times \langle 2, 2, 6t\rangle\|}{\sqrt{8t^2 + 9t^4}} = \frac{\|\langle 6t^2, -6t^2, 0\rangle\|}{t\sqrt{8 + 9t^2}} = \frac{6t^2\|\langle 1, -1, 0\rangle\|}{t\sqrt{8 + 9t^2}}$$

$$\text{so, } a_N = \frac{6t\sqrt{2}}{\sqrt{8 + 9t^2}}$$

Second Newton's Law: If at time t a force $\mathbf{F}(t)$ acts on an object of mass m producing an acceleration $\mathbf{a}(t)$, then $\mathbf{F}(t) = m\mathbf{a}(t)$.

Ex3. Projectile Motion.

$$\begin{cases} v(0) = \vec{v}_0 = \langle |v_0|\cos\alpha, |v_0|\sin\alpha \rangle \\ r(0) = \vec{0} \end{cases}$$



A particle is fired with angle of elevation α and initial velocity $\mathbf{v}_0$ (See figure on the left.) Assuming that the air resistance is negligible and the only external force is due to gravity, find the position $\mathbf{r}(t)$ of the projectile. What value of α maximizes the range(the horizontal distance traveled)?

$\vec{a} = \langle 0, -g \rangle \qquad g = 9.8\ m/s^2$

$\Rightarrow v(t) = \langle 0, -gt \rangle + \vec{c}$ but $\underbrace{v(0)}_{\langle 0,0\rangle + \vec{c}} = \vec{v}_0$

$\langle 0, 0 \rangle + \vec{c} = \vec{v}_0 \Rightarrow \vec{c} = \vec{v}_0$

So, $v(t) = \langle 0, -gt \rangle + \vec{v}_0$

then $r(t) = \langle 0, \frac{-gt^2}{2} \rangle + t\vec{v}_0 + \vec{D}$ but $r(0) = \vec{0}$

$\langle 0, 0 \rangle + 0\vec{v}_0 + \vec{D} = \vec{0}$

$\Rightarrow \vec{D} = \vec{0}$

so, $r(t) = \langle 0, -g\frac{t^2}{2} \rangle + t\vec{v}_0 + \vec{0}$

$$\boxed{r(t) = \langle t|v_0|\cos\alpha, -g\frac{t^2}{2} + t|v_0|\sin\alpha \rangle}$$

we need $\frac{-gt^2}{2} + t|v_0|\sin\alpha = 0 \Rightarrow t\left(-\frac{gt}{2} + |v_0|\sin\alpha\right) = 0$

$t = 0$, $t = \frac{2|v_0|\sin\alpha}{g}$

then $d = t|v_0|\cos\alpha = \frac{2|v_0|^2\sin\alpha\cos\alpha}{g} = \frac{|v_0|^2\sin(2\alpha)}{g}$

we want $\sin(2\alpha) = 1 \Rightarrow 2\alpha = \frac{\pi}{2} \Rightarrow \boxed{\alpha = \frac{\pi}{4}}$